M.Sc Sem III, paper XI.

(Normed spaces) Important Examples.

Example - Let N be a non-zero normed linear space and let $S = \{x \in N : \|x\| \leq \}$ be a linear subspace of N. Prove that N is a Banach space $\Leftrightarrow$ S is complete.

Solution:- First assume that N is a Banach space. Let $\langle x_n \rangle$ be a Cauchy sequence in S so that $\|x_n\| = 1$ for every n. Since $S \subset N$, $\langle x_n \rangle$ is also a Cauchy sequence in N. Since N is complete, there exists $x \in N$ such that $x_n \to x$. Since norm is a continuous function, we have

$$x_n \to x \Rightarrow \|x_n\| \to \|x\|. \text{ Thus}$$

$$\|x\| = \lim_{n \to \infty} \|x_n\| \leq 1 \quad [\because \|x_n\| \leq 1 \, \forall n]$$

it follows that $x \in S$ and so S is complete.

Conversely, let S be complete. To prove that N is Banach space. Since N is a normed linear space, we need only prove that N is complete.

Let $\langle y_n \rangle$ be any Cauchy sequence in N. Then $\|y_m - y_n\| \to 0$ as $m, n \to \infty$.

For each n define $x_n = \frac{y_n}{\|y_n\|}$. Then

$$\|x_n\| = \left\| \frac{y_n}{\|y_n\|} \right\| = \frac{1}{\|y_n\|} \cdot \|y_n\| = 1 \text{ and so}$$

$x_n \in S$.

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Expt. No. / Name We shall show that $\langle x_n \rangle$ is a Cauchy sequence in S. We have Date / /

$$\|x_m - x_n\| = \left\| \frac{y_m}{\|y_m\|} - \frac{y_n}{\|y_n\|} \right\|$$

$$= \left\| \frac{y_m}{\|y_m\|} - \left\| \frac{y_n}{\|y_m\|} + \frac{y_n}{\|y_m\|} - \frac{y_n}{\|y_n\|} \right\|$$

$$\leq \left\| \frac{y_m}{\|y_m\|} - \frac{y_n}{\|y_m\|} \right\| + \left\| \frac{y_n}{\|y_m\|} - \frac{y_n}{\|y_n\|} \right\|$$

$$= \left\| \frac{y_m}{\|y_m\|} (y_m - y_n) \right\| + \left\| \left(\frac{1}{\|y_m\|} - \frac{1}{\|y_n\|} \right) y_n \right\|$$

$$= \frac{1}{\|y_m\|} \|y_m - y_n\| + \left| \frac{1}{\|y_m\|} - \frac{1}{\|y_n\|} \right| \|y_n\|$$

$$[\because \|\alpha x\| = |\alpha| \, \|x\|]$$

$$= \frac{\|y_m - y_n\|}{\|y_m\|} + \frac{\big| \|y_n\| - \|y_m\| \big|}{\|y_m\|}$$

$$\leq \frac{\|y_m - y_n\|}{\|y_m\|} + \frac{\|y_n - y_m\|}{\|y_m\|}$$

$$= \frac{2\|y_m - y_n\|}{\|y_m\|} \to 0 \text{ as } m, n \to \infty.$$

Hence $\langle x_n \rangle$ is a Cauchy sequence in S. Since S is complete, there exists $x \in S$ such that

$$x_n \to x \Rightarrow \frac{y_n}{\|y_n\|} \to x \quad \text{---} (1)$$

Also $\big| \|y_n\| - \|y_m\| \big| \leq \|y_n - y_m\| \to 0$ and so $\langle \|y_n\| \rangle$ is a Cauchy sequence of real numbers. Since R is complete, $\|y_n\| \to \alpha \in R$ and consequently by (1), we have $y_n \to \alpha x \in N$. Hence N is complete. Proved.

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Ex-(02) Let N and N' be normed linear spaces and let S be a linear transformation of N into N'. Then T is continuous either at every point of N or at no point of N.

Solution:- Let x_1 and x_2 be any two points of N and suppose T is continuous at x_1. Then to each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x - x_1\| < \delta = \|T(x) - T(x_1) < \epsilon$$

Now $\|x - x_2\| < \delta \Rightarrow \|(x + x_1 - x_2) - x_1\| < \delta$

$$\Rightarrow \|T(x + x_1 - x_2) - T(x_1)\| < \epsilon$$

$$\Rightarrow \|T(x) + T(x_1) - T(x_2) - T(x_1)\| < \epsilon$$

(By linearity of T)

$$\Rightarrow \|T(x) - T(x_2)\| < \epsilon$$

This shows that T is continuous at x_2 as well. Since x_1, x_2 are arbitrary points, we have shown that if T is continuous at a particular point, then it is continuous at all points.

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